TECHNICAL REPORT
67-74-OSD

ON MEMBRANE FREQUENCIES FOR SPHERICAL SHELL VIBRATIONS

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CONTENTS

ABSTRACT

In this paper the membrane solutions for non-symmetric vibration of a spherical dome are studied. The solutions are written in a convenient form, and it is shown how these reduce to the familiar membrane and inextensional solutions in the static limit. This information enables one to perceive several difficulties in finding inextensional frequencies by numerical means and to suggest ways around these difficulties. Also membrane frequencies are found for circumferential wave numbers up through eight by direct calculation and compared with several different approximations. One type of approximation, due to Jeffreys and Jeffreys, gives quite good results. It is seen that earlier work agrees well with these results except for a few frequencies which are not found at all in the present calculations.

1. Introduction

In studying the behavior of tents and parachutes it is pertinent to study the vibrations of thin shells. In this paper we shall focus our attention on the non-symmetric vibration of thin spherical shells. Within the classical thin-shell theory the governing system of equations is of eighth order, and solutions in the form of Associated Legendre Functions have been found by Kalnins and Wilkinson[1] and Prasad[2]. Four of these solutions are of bending type and can often be approximated with the aid of the asymptotic methods used by the writer[3-5]. Although we shall make some remarks about these solutions, our main concern here is with the remaining four solutions, which are of membrane type.

We shall first write down these four solutions in various forms that are useful and shall describe some approximations that are applicable in several limiting cases. Then in Section 3 we shall present formulas showing how these dynamic solutions reduce to the familiar static membrane and inextensional solutions as the dimensionless frequency, Ω, tends to zero. It is helpful to have this information, in particular, to know what linear combinations of the Legendre Function solutions approach the inextensional solutions as $\Omega \to 0$, before one tries to deduce inextensional frequencies and modes from the Legendre Function solutions. This new information should help in evading the difficulty that has been experienced (see Hwang[6]) in making these computations. Section 4 contains results about membrane natural frequencies for hemispheres with free-edges. These results are compared with each other and also with the earlier calculations of Naghdi and Kalnins[7]. The results are discussed in Section 5.

1. [illegible] of the Membrane System

The four membrane solutions will be designated as $\underset{\sim}{S}_j^m$ and have been given in terms of Associated Legendre Functions by Naghdi and Kalnins[7]. We shall use slightly modified versions of their solutions, which are convenient in studying the limit as $\Omega \to 0$.

$$\underset{\sim}{S}_1^m: \; w_1^m = Z_n^m(\cos\phi) \qquad\qquad \underset{\sim}{S}_2^m: \; w_2^m = Q_n^m(\cos\phi)$$

$$u_1^m = A\, dw_1^m/d\phi \qquad\qquad u_2^m = A\, dw_2^m/d\phi$$

$$v_1^m = -mAw_1^m \csc\phi \qquad\qquad v_2^m = -mAw_2^m \csc\phi$$

$$\underset{\sim}{S}_3^m: \; u_3^m = -mA\csc\phi\, Z_k^m(\cos\phi) \tag{1}$$

$$v_3^m = A\, dZ_k^m/d\phi, \quad w_3^m = 0$$

$$\underset{\sim}{S}_4: \; u_4^m = -mA\csc\phi\, Q_k^m(\cos\phi)$$

$$v_4^m = A\, dQ_k^m(\cos\phi)/d\phi, \quad w_4^m = 0$$

Here m is the circumferential wave number (a non-negative integer for shells of revolution) and

$$A = (1 - \Omega^2) / [1 + (1 + \nu)\Omega^2]$$

$$(n + \tfrac{1}{2})^2 = (3/2)^2 + (1 + \nu)\Omega^2 \left[3 - (1 - \nu)\Omega^2\right] / (1 - \Omega^2) \tag{2}$$

$$(k + \tfrac{1}{2})^2 = (3/2)^2 + 2(1 + \nu)\Omega^2 \tag{3}$$

where ν is Poisson's Ratio. The functions Z_n^m and Z_n^m are related to the Associated Legendre Functions by e.g.

$$Z_n^m(\cos\phi) = (-2)^m\, m!\, \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\, P_n^m(\cos\phi)$$

$$= \sin^m\phi \;\; F[1+m+n,\, m-n;\, m+1;\, \tfrac{1}{2}(1-\cos\phi)] \tag{4}$$

where F (...) denotes the hypergeometric function. We shall also use the representation

$$Z_n^m = ([illegible])^m \, F\left[-n, n+1; m+1; \tfrac{1}{2}(1-\cos\phi)\right] \qquad (5)$$

which may be derived from the definition by a familiar property of the hypergeometric function.

A useful pair of asymptotic formulas for $P_n^m(\cos\phi)$ when n and m are both large and

$$m \neq n \sin\phi \qquad (6)$$

has been given by Jeffreys and Jeffreys[8]. From them we derive the following asymptotic formulas for Z_n^m:

$m > n\sin\phi$:

$$Z_n^m \sim Y_n^m (M - n\cos\phi)^{-n-\frac{1}{2}} (M + m\cos\phi)^{-m} M^{-\frac{1}{2}} \sin^m\phi \qquad (7)$$

where

$$M = M_n = (m^2 - n^2\sin^2\phi)^{\frac{1}{2}}$$

$$Y_n^m = 2^m \frac{m!\, n!}{(m+n)!} (2\pi)^{-\frac{1}{2}} \left(\frac{m^2-n^2}{n} \right)^{n+\frac{1}{2}} (m+n)^m$$

$m < n\sin\phi$:

$$Z_n^m \sim D_n^m N^{-\frac{1}{2}} \sin \xi_n^m \qquad (8)$$

where

$$N = N_n = (n^2\sin^2\phi - m^2)^{\frac{1}{2}}$$

$$D_n^m = 2^m \frac{m!\, n!}{(m+n)!} (2\pi)^{-\frac{1}{2}} \left(1 - \frac{m^2}{n^2}\right)^{\frac{1}{2}(n+\frac{1}{2})} \left(\frac{n+m}{n-m}\right)^{m/2}$$

$$\xi_n^m = (n+\tfrac{1}{2})\beta - m\alpha + \tfrac{1}{4}\pi$$

$$\beta = \text{arc cos} \left[n \cos\phi \,/\, (n^2 - m^2)^{\frac{1}{2}}\right], \qquad 0 \leq \beta \leq \pi$$

$$\alpha = \text{arc cos} \left[m \cot\phi \,/\, (n^2 - m^2)^{\frac{1}{2}}\right], \qquad 0 \leq \alpha \leq \pi$$

There are several points of interest about these solutions that we can observe without further effort. First, the writer has previously observed that in the axisymmetric case ($m=0$) the torsionless membrane solutions, $\underset{\sim}{S}_1^m$ and $\underset{\sim}{S}_3^m$, are not necessarily accurate approximations to solutions of the complete (i.e., membrane plus bending) system when $\Omega = 1$. This defect of the membrane approximation was also shown to be symptomatic of the occurrence of a transition point (of infinite order) in the asymptotic approximations to the bending solutions when $\Omega = 1$. The singularity in (2) when $\Omega=1$ is evidence that this difficulty persists in the non-symmetric case. However, the analogous equation for k, (3), contains no such singularity, and we conclude that the torsional membrane solutions $\underset{\sim}{S}_2^m$ and $\underset{\sim}{S}_4^m$, are free of this defect at $\Omega=1$.

Second the asymptotic formulas (7) and (8) for Z_n^m are valid when n and m are both large and are of course equally applicable to the function Z_k^m. These asymptotic representations suffer from a transition point at $\phi=\phi_t$, where

$$\sin \phi_t = m/n \tag{9}$$

It is worth dwelling on the differences between this transition point and the one described above, where $\Omega = 1$.

The most obvious difference between the two kinds of transition points is this. The transition point for $\Omega=1$ shows itself as a singularity in the relation (2) between n and Ω and does not involve m, whereas the present transition point is determined by the relation (9) between m and n and does not show itself as a singularity of any

... we compare the membrane system of equations with the complete system for a sphere we see[5] that (i) the membrane system has a singularity when Ω=1 but the complete system does not, and (ii) neither system has a singularity at points given by (9). We conclude that the singularity in the membrane equations for Ω=1 implies that some of the membrane solutions are then poor approximations to complete solutions. The absence of such a singularity at $\phi=\phi_t$ means that the present transition point does not significantly affect the accuracy of the membrane solutions as approximations to complete solutions but merely affects the accuracy of the asymptotic formulas, (7) and (8), as approximations to the membrane solutions. Alternatively, we may say that, when Ω=1, bending effects intrude into the membrane equations, or that there is some exchange of bending and stretching energies, but bending does not affect what happens at $\phi=\phi_t$. The shortcomings of the approximations (7) and (8) near ϕ_t can be overcome entirely within the scope of the membrane theory, without considering bending effects.

3. Behavior of Membrane Solutions as Ω>0.

We consider first the passage to the static limit, $\Omega \to 0$. In this case $n \to 1$ and $k \to 1$ and

$$Z_1^m = \sin^m\phi \; F\left[m+2,\ m-1;\ m+1;\ \tfrac{1}{2}(1-\cos\phi)\right]$$

$$= \left[2^m / (1+m)\right] \; (m+\cos\phi)\ \tan^m \tfrac{1}{2}\phi$$

$$Q_1^m = (-\sin\phi)^m \; d^m\, Q_1^o\,(x) / dx^m \quad , \quad x = \cos\phi$$

$$Q_1^o\,(x) = \tfrac{1}{2}\, x \ln\ (1+x)/(1-x) \;-\; 1$$

Love[9] gives the solutions X_j^m (j=1,2,3,4) to the statical membrane solutions, with the following displacements:

$m = 0:\quad \underset{\sim}{X}_1^{\,0}: w_1^{\,0} = \cos\phi \qquad \underset{\sim}{X}_2^{\,0}: w_2^{\,0} = -\cos\phi\,\ln(\tan\tfrac{1}{2}\phi) - 1$

$\qquad u_1^{\,0} = -\sin\phi \qquad\qquad u_2^{\,0} = \sin\phi\,\ln(\tan\tfrac{1}{2}\phi) - \cot\phi$

$\qquad v_1^{\,0} = 0 \qquad\qquad v_2^{\,0} = 0$

$\underset{\sim}{X}_3^{\,0}: w_3^{\,0} = u_3^{\,0} = 0 \qquad \underset{\sim}{X}_4^{\,0}: w_4^{\,0} = u_4^{\,0} = 0$

$\qquad v_3^{\,0} = -\sin\phi \qquad\qquad v_4^{\,0} = \sin\phi\,\ln(\tan\tfrac{1}{2}\phi) - \cot\phi$

$m = 1:\quad \underset{\sim}{X}_1^{\,1}: w_1^{\,1} = -\sin\phi \qquad \underset{\sim}{X}_2^{\,1}: w_2^{\,1} = \sin\phi$

$\qquad u_1^{\,1} = v_1^{\,1} = 1 - \cos\phi \qquad u_2^{\,1} = -v_2^{\,1} = 1 + \cos\phi$

$\underset{\sim}{X}_3^{\,1}: w_3^{\,1} = \cot\phi - \sin\phi\,\ln(\tan\tfrac{1}{2}\phi)$

$\qquad u_3^{\,1} = v_3^{\,1} = (1-\cos\phi)\,\ln(\tan\tfrac{1}{2}\phi) - [(2-\cos\phi)/(1-\cos\phi)]$

$\underset{\sim}{X}_4^{\,1}: w_4^{\,1} = -\cot\phi + \sin\phi\,\ln(\tan\tfrac{1}{2}\phi)$

$\qquad u_4^{\,1} = -v_4^{\,1} = (1+\cos\phi)\,\ln(\tan\tfrac{1}{2}\phi) + [(2+\cos\phi)/(1+\cos\phi)]$

$m \geq 2:\quad \underset{\sim}{X}_1^{\,m}: w_1^{\,m} = -(m+\cos\phi)\tan^m\tfrac{1}{2}\phi \qquad \underset{\sim}{X}_2^{\,m}: w_2^{\,m} = (m-\cos\phi)\cot^m\tfrac{1}{2}\phi$

$\qquad u_1^{\,m} = v_1^{\,m} = \sin\phi\tan^m\tfrac{1}{2}\phi \qquad u_2^{\,m} = -v_2^{\,m} = \sin\phi\cot^m\tfrac{1}{2}\phi$

$\underset{\sim}{X}_3^{\,m}: w_3^{\,m} = \left[-4\csc^2\phi + (m-\cos\phi)G_3(\phi)\right]\tan^m\tfrac{1}{2}\phi$

$u_3^{\,m} = -v_3^{\,m} = G_3(\phi)\,\sin\phi\tan^m\tfrac{1}{2}\phi$

$G_3(\phi) = (2/m) + (m+1)^{-1}\tan^2\tfrac{1}{2}\phi + (m-1)^{-1}\cot^2\tfrac{1}{2}\phi$

$\underset{\sim}{X}_4^{\,m}: w_4^{\,m} = \left[4\csc^2\phi - (m+\cos\phi)G_4(\phi)\right]\cot^m\tfrac{1}{2}\phi$

$u_4^{\,m} = v_4^{\,m} = G_4(\phi)\,\sin\phi\cot^m\tfrac{1}{2}\phi$

$G_4(\phi) = (2/m) + (m-1)^{-1}\cot^2\tfrac{1}{2}\phi + (m-1)^{-1}\tan^2\tfrac{1}{2}\phi$

... that the X_1^m and X_2^m are those corresponding to rigid body [illegible] for m=1 and inextensional motions for $m\geq 2$, and the solutions [illegible] are those for which the stress resultants $n_{\phi\phi}$, $n_{\theta\theta}$ and $n_{\phi\theta}$ [illegible] vanish identically. The relations between X_j^m and the limits [illegible] of the vibrational solutions S_j^m are found by comparing behaviors at $\phi=0$ and $\phi=\frac{1}{2}\pi$. The results are

$$S_j^0 \to X_j^0 , \qquad j=1,2,3,4 \tag{10}$$

$$\left.\begin{aligned} -(S_1^1 + S_3^1) &\to X_1^1, \qquad S_1^1 - S_3^1 \to X_2^1 \\ -(S_2^1 + S_4^1) &\to X_3^1 , \qquad S_2^1 - S_4^1 \to X_4^1 \end{aligned}\right\} \tag{11}$$

and for $m \geq 2$

$$\left.\begin{aligned} & -(1+m)\, 2^{-m} (S_1^m + S_3^m) \to X_1^m \\ & B_2 (S_1^m - S_3^m) + \frac{2(-1)^m}{(m-2)!} (S_2^m - S_4^m) \to X_2^m \\ & \frac{2^{-m+1}}{m(m-1)} (S_1^m - S_3^m) \to X_3^m \\ & B_4 (S_1^m + S_3^m) + \frac{4(-1)^{m+1}}{(m+1)!} (S_2^m + S_4^m) \to X_4^m \end{aligned}\right\} \tag{12}$$

where

$$\begin{aligned} B_2 &= 2^{-m}(m^{-1} + 1)\Big[m + \\ & \frac{(-2)^m \pi^{\frac{1}{2}}}{(m-2)!} \frac{([illegible])!}{([illegible])!} \sin\left\{\tfrac{1}{2}\pi (m+1)\right\}\Big] \\ B_4 &= 2^{-m+1} m^{-1} \Big[(m-1)^{-1} + \\ & \frac{(-2)^m \pi^{\frac{1}{2}}}{m!} \frac{([illegible])!}{([illegible])!} \sin\left\{\tfrac{1}{2}\pi (m+1)\right\}\Big] \end{aligned} \tag{13}$$

4. Natural Frequencies for a Free-Edged Hemisphere

In this section we shall use the formulas of Section 3 to obtain estimates of the membrane natural frequencies for a hemispherical

[illegible] the frequency condition has been written by [illegible], and may be written

$$[illegible] - ([illegible])^2 = 0 \qquad [illegible]$$

[illegible]:

$$S_n^m = \left.\frac{dZ_n^m(\cos\phi)/d\phi}{Z_n^m(\cos\phi)}\right|_{\pi/2}, \qquad S_k^m = \left.\frac{dZ_k^m(\cos\phi)/d\phi}{Z_k^m(\cos\phi)}\right|_{\pi/2}$$

$$[illegible] = 1 + (1 + \nu)\Omega^2$$

If the representation (5) is used, the frequency condition can be written

$F_{n} = F(-n, n+1; m+1; \;)$, $F_{kn} = F(-n+1, n+2; m+2; \;)$ (1.

... are defined ... larly.

... (15) was made the basis of a trial-and-error calculation ... frequencies for $m \leq 8$. The results are accurate to three ... figures and are presented in Table 1 and Figure 1. For ... on we also present the results given by Naghdi and Kalnins[7].

One interesting result is very easily derived from (15). If $n \gg k$ and $m \gg |n|$, all the hypergeometric functions in (15) are approximately unity, and (15) reduces approximately to

$$n(n+1) + k(k+1) - 4\left[1 + (1+\nu)\Omega^2\right] = 0$$

Combining this with (2) and (3), we obtain

$$\Omega^4\left[1 + (1+\nu)\Omega^2\right] / (1 - \Omega^2) = 0$$

a relation which cannot be satisfied for real, non-zero values of Ω. We conclude that no membrane frequencies (i.e., no natural frequencies with modes of membrane type) can be found when $n \gg \Omega$. Since this has been derived solely on the basis of the membrane solutions, we must observe that it does not preclude the occurrence of bending and inextensional frequencies, nor does it rule out the transitional frequencies near $\Omega = 1$.

It is also helpful to see what is obtained if we combine the approximate formulas (7) and (8) for both n and k with the frequency relation (14). The asymptotic formulas for $dZ_n^m/d\varphi$ may be obtained either by differentiating the asymptotic relation for Z_n^m (which is ... in this case though not in general), or by using the

relation

$$dP_n^m/d\phi = (n-m+1)\csc\phi\, P_{n+1}^m - (n+1)\cos\phi\, P_n^m$$

together with the asymptotic relation for Z_n^m. The frequency equation assumes different forms in three regions, as follows:

(i) $m > k > n$

$$(m - Hm^{-1})^2 - (M_n - \tfrac{1}{2}nM_n^{-1})(M_k - \tfrac{1}{2}kM_k^{-1}) = 0 \tag{17}$$

(ii) $k > m > n$

$$(m - Hm^{-1})^2 \tan\tfrac{1}{2}\pi(k-m+1)$$
$$- (M_n - \tfrac{1}{2}nM_n^{-1})(N_k - \tfrac{1}{2}kN_k^{-1}) = 0 \tag{18}$$

(iii) $k > n > m$

$$(m - Hm^{-1})^2 \tan\tfrac{1}{2}\pi(k-m+1)\tan\tfrac{1}{2}\pi(n-m+1)$$
$$- (N_n - \tfrac{1}{2}nN_n^{-1})(N_k - \tfrac{1}{2}kN_k^{-1}) = 0 \tag{19}$$

Natural fr··· ·encies may be estimated easily in the various regions with the aid of these formulas. The results of these computations are compared with the calculations based on the accurate frequency equation (15) in Table I.

A further simplification is possible when $\Omega^2 \gg m^2$ or

$$(k+\tfrac{1}{2})^2 > (n+\tfrac{1}{2})^2 \gg m^2 ,$$

in which case (19) becomes

$$\tan\tfrac{1}{2}\pi(k-m+1)\ \tan\tfrac{1}{2}\pi(n-m+1) = O(m^2\Omega^{-2})$$

whence

$$k + \tfrac{1}{2} \approx 2J - m - \tfrac{1}{2} \tag{20}$$

$$n + \tfrac{1}{2} \approx 2L - m - \tfrac{1}{2} \tag{21}$$

where J and L are sufficiently large positive integers. When $\Omega^2 \gg 1$, we have from (2) and (3)

$$n + \tfrac{1}{2} \approx \Omega(1 - \nu^2)^{1/2}$$

$$k + \tfrac{1}{2} \approx \Omega[2(1 + \nu)]^{1/2}$$

Combining these with (20) and (21) we find two families of natural frequencies, given by

$$\Omega \approx (2J + m - \tfrac{1}{2})(1 - \nu^2)^{-1/2} \qquad (22)$$

$$\Omega \approx (2L + m - \tfrac{1}{2})[2(1 + \nu)]^{-1/2} \qquad (23)$$

The first of these families is associated with stretching (torsionless) membrane modes, the second with torsional modes. The frequency for each branch depends linearly on m with different slopes for the two families. These predictions are shown in Table I and Figure 1.

5. Discussion

We shall comment first on the behavior of the solutions as $\Omega \to 0$ and then on the natural frequency calculations.

The formulas (10)-(13) show clearly that in the limit as $\Omega \to 0$ the dynamic membrane solutions in the form of Legendre Functions are equivalent to the familiar static membrane and inextensional solutions. We find that, as $\Omega \to 0$, two linear combinations of the dynamic solutions tend toward the static, inextensional solutions and two other combinations tend to the static membrane solutions. These linear combinations are given in (10)-(13).

We shall now point out several of the pitfalls that one may encounter in trying to calculate the inextensional frequencies for a free-edged dome from the general solution. The general solution with

[illegible] ential wave number m is (for a dome) expressed in terms of [illegible] ciated Legendre Functions in the form

$$w^m = C_n P_n{}^m(\cos\phi) + C_k P_k{}^m(\cos\phi)$$
$$+ C_{b_1} P_{b_1}(\cos\phi) + C_{b_2} P_{b_2}(\cos\phi) \qquad ([illegible])$$

with derived expressions for the other variables. Here n and k are the degrees of the Legendre Functions connected with the membrane solutions, and b_1 and b_2 are the degrees connected with the bending solutions. These solutions were given explicitly by Kalnins and Wilkinson[1].

Now the inextensional frequencies are very low, i.e.,

$$\Omega^2 = 0\,(\,h^2 / R^2\,) << 1$$

where h is the thickness and R the radius of the shell. We have previously pointed out[5] that, when Ω is this small, the bending solutions are nearly statical and are both of edge-effect type. Also, we see from (2) and (3) that

$$n \approx k \approx 1$$

It is an inconvenient property of the Associated Legendre Function that, e.g., $P_n{}^m(\cos\phi)$ vanishes identically when m is an integer and $m > n$. Since inextensional solutions occur only when $m \geqslant 2$, we see that $P_n{}^m(\cos\phi)$ and $P_k{}^m(\cos\phi)$ will be very small for the inextensional modes. This may make it hard to evaluate C_n and C_k numerically. This difficulty can be evaded by using the solutions $Z_n{}^m$, $Z_k{}^m$, which do not become small when n and k are near unity, i.e., we may write the general solution as

$$w^m = C_n' \; Z_n{}^m(\cos\phi) + C_k' \; Z_k{}^m(\cos\phi)$$
$$+ C_{b_1} P_{b_1}(\cos\phi) + C_{b_2} P_{b_2}{}^m(\cos\phi) \qquad (25)$$

However, we are not out of the woods yet, for it is only a very special linear combination of Z_n^m and Z_k^m that leads to an inextensional solution. If we rewrite (25) as

$$w^m = \tfrac{1}{2}(C_n' + C_k')(Z_n^m + Z_k^m) + \tfrac{1}{2}(C_n' - C_k')(Z_n^m - Z_k^m)$$
$$+ C_{b_1} P_{b_1}(\cos\phi) + C_{b_2} P_{b_2}(\cos\phi)$$

we see from (12) that the first solution becomes inextensional as $\Omega \to 0$, but the second solution is a membrane solution. In order to obtain a mode which is predominantly inextensional we must have

$$\frac{C_n' - C_k'}{C_n' + C_k'} \ll 1$$

It is plain that, if the constants are evaluated numerically with insufficient accuracy, so that C_n' and C_k' are not quite equal, we may be accidentally introducing a little bit of the membrane solution. Even a little trace of the membrane solution is enough to destroy the inextensional property of the solution and prevent one from obtaining inextensional frequencies. It is possible that this difficulty can be overcome by careful calculation. If not, it may be necessary to set $C_n' = C_k'$ throughout the calculation, temporarily discarding one membrane boundary condition. After an inextensional frequency has been found, the discarded boundary condition can be used to calculate the difference $C_n' - C_k'$, which is initially taken as exactly zero.

The results for membrane natural frequencies of a free-edge hemisphere are given Table I and Figure 1. The gross features are these:

(i) When $m^2 \gg \Omega^2$, there are no membrane frequencies.

(ii) When $\Omega^2 \gg m^2$, the simple asympstotic estimates (22) and (23) are fairly reliable. These predict two families of frequencies which

depend linearly on m, the slopes of the two families being different. These predictions are less accurate near points where two frequency lines cross than elsewhere, and the errors in the asymptotic estimates change sign at such points.

(iii) If Ω^2 and m^2 are of comparable size, the picture is more complicated. When Ω and m are both large, the most important analytical features are the n- and k- transition lines, i.e., the locus of points for which $n=a$ and $k=a$ respectively. These are shown in Figure 1. Below the k-line the approximate frequency condition, (17), involves no oscillatory functions, and there is merely a single frequency for each m. Between the n-and k-transition lines the frequency condition, (18), contains an oscillatory function of k, indicating that there is one family of frequencies, namely those of torsional type. Above the n-line two families of frequencies are found, corresponding to the two oscillatory functions that occur in the frequency condition (19).

Interesting information is revealed by comparisons among the various approximations given in Table I. First, we see that the approximate frequency equations, (17)-(19), give generally good agreement with the more accurate frequency equation, (15). The accuracy is poorest, of course, near the n- and k- transition lines. The simple asymptotic estimates, (22) and (23), are not as accurate as (17) and (19) but are tolerably good when $\Omega^2 \gg m^2$. They are very poor near or below the n-transition line.

But, the calculations based on (15) do not show complete agreement with the previous results of Naghdi and Kalnins[7]. For m=1 the agreement is very good. For m=2 and 3 there is good agreement for some frequencies, but the present calculations do not show certain frequencies that were found earlier. Separate hand calculations were made for each of these questionable frequencies. In no case was a frequency found. Moreover, if we plot these doubtful frequencies on Figure 2, we see that they do not fit well with the pattern of the remaining frequencies. We conclude, therefore, that the frequencies marked with a question mark in Table 1 are spurious.

In closing we may remark that these membrane frequencies are all high compared with the inextensional frequencies. This does not necessarily mean that they are unimportant, however. Their importance in a problem of time-dependent loading depends on whether their mode makes a significant contribution to the response of the shell. This in turn depends on the kind of applied load and its spatial distribution. It is relevant that the modes[5] associated with most of these frequencies involve much more motion in the shell surface than normal to it. Therefore, if experiments on shell vibration are made in which only motion normal to the surface is measured, it will be very easy to miss these modes and conclude (erroneously) that they are unimportant.

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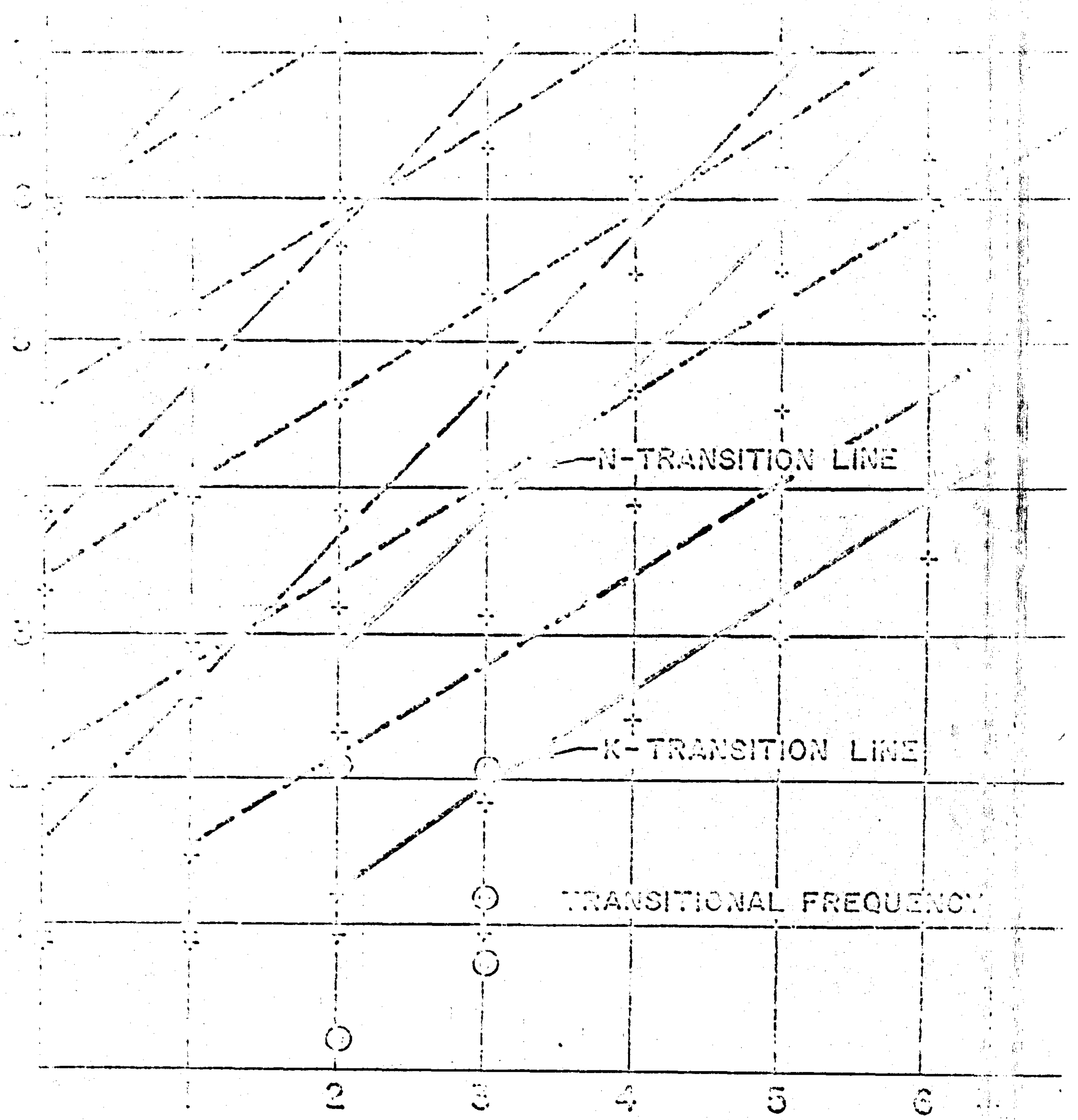
[illegible] 1. [illegible] for various [illegible] wave numbers, m, [illegible] according to membrane theory.

	Numerical[a]	Asymptotic[b]	Simple[c] Asymptotic	Ref. 7
m = 1	.876			.876
	.950			.950
	1.47			1.47
	2.55			2.56
	2.92			2.92
	3.92	3.92	4.0[illegible]	
	4.80	4.81	4.72[t]	
	5.21	5.20	5.27[t]	
	6.43	6.44	6.51[t]	
	6.89	6.88	6.81[s]	
m = 2				.215 (?)
	.916			.922
	1.21			1.21
				2.08 (?)
	2.31			2.31
	3.18	3.26	3.41[t]	
	3.84	3.83	3.67[s]	
	4.60	4.57	4.65[t]	
	5.66	5.69	5.77[s]	
	5.99	5.97	5.89[t]	
	7.06	7.06	7.13[t]	
m = 3				.740 (?)
	.943			.943
				1.20 (?)
	1.84			1.83
				2.07 (?)
	3.13	3.09	2.79[t]	
	3.88	3.94	4.03[t]	
	4.70	4.75	4.72[s]	
	5.33	5.28	5.27[t]	
	6.35	6.33	6.51[t]	
	6.93	6.92	6.81[s]	
m = 4	2.41			
	3.87	3.86		
	4.67	4.57	4.65[t]	
	5.47	5.58	5.77[s]	
	6.15	6.08	5.89[t]	
	7.04	7.05	7.13[t]	

	Numerical[a]	Asymptotic[b]	Simple Asymptotic[c]	Ref[7]
m = 5	[illegible]	2.81		
	4.57	4.55		
	5.49	[illegible]	5.27[t]	
	6.21	6.35	6.51[t]	
	6.98	6.95	6.81[s]	
m = 6	3.54	3.41		
	5.19	5.19		
	6.27	6.23		
	6.99	7.09	7.13[t]	
m = 7	4.10	3.98		
	5.82	5.81		
	7.01	6.98		
m = 8	4.65	4.55		
	6.44	6.42		

a based on Equation (15)
b based on Equations (17) - (19)
c based on Equations (22), (23)
t associated with torsional node
s associated with stretching node



Dimensionless Frequency, $\Omega = \omega R(\rho/E)^{\frac{1}{2}}$
versus circumferential wave number, m, for a
free edged hemisphere according to membrane theory.

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(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) U. S. Army Natick Laboratories Natick, Massachusetts 01760	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

ON MEMBRANE FREQUENCIES FOR SPHERICAL SHELL VIBRATIONS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (Last name, first name, initial)

Ross, Edward W., Jr.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1967	14	9

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 67-74-OSD 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

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In this paper the membrane solutions for non-symmetric vibration of a spherical dome are studied. The solutions are written in a convenient form, and it is shown how these reduce to the familiar membrane and inextensional solutions in the static limit. This information enables one to perceive several difficulties in finding inextensional frequencies by numerical means and to suggest ways around these difficulties. Also membrane frequencies are found for circumferential wave numbers up through eight by direct calculation and compared with several different approximations. One type of approximation, due to Jeffreys and Jeffreys, gives quite good results. It is seen that earlier work agrees well with these results except for a few frequencies which are not found at all in the present calculations.

DD FORM 1 JAN 64 1473

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Continuum mechanics	8					
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